Swap Stability in Refugee Housing: A Story about Anonymous Preferences

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Abstract. In refugee housing, we are given a topology modelled as an undirected graph, a set of inhabitants assigned to some vertices or houses of the topology together with their approval preferences specifying the number of refugees they admit in their neighbourhood, as well as a number R of refugees we need to accommodate in the empty houses of the topology. In the associated computational problem, we are looking for a housing such that no agent is able to swap its location with other agent to improve the utilities of both of them.

We show conditions under which such swap-stable housing always exists and can be found in polynomial time. Then, we turn our attention to the efficiency of swap-stable solutions using the notion of (utilitarian) social welfare. In contrast to the polynomial-time algorithm that finds a swap-stable housing, we prove that deciding whether there exists a swap-stable housing with social welfare at least ξ is NP-hard. For the optimisation variant, we show that unless $\mathsf{P} = \mathsf{NP}$, there is no constant-factor polynomial-time approximation algorithm. Finally, we study how social welfare may be decreased due to swap-stability requirement using the notion of Price of Stability.

1 Introduction

Refugee housing is an emerging topic in today's world. The Data Portal[1] of the United Nations High Commissioner for Refugees (UNHCR) offers an important overview of all situations resulting in a significant number of refugees. Events described as *featured situations* in that report, which include the Russian war against Ukraine and armed conflicts and political instability in Myanmar and Sudan, involve at least 10 additional millions of people were forced to leave their homes in the last two years.

Following the dire need to help the refugee situation, the algorithmic aspects of their accommodation has also received attention in the computer science literature. The main line of research, initiated by Delacretaz et al. [6], involves finding a fair and even redistribution of refugees between different countries or regions. The standard model here is the double-sided matching market, including features and constraints with locations on one side and refugees on the other side. Many authors have further investigated this model from the perspective of mechanism design [2], computational complexity [4, 8], or machine learning techniques [5].

Recently, Knop and Schierreich [7] introduced a model for refugee housing at the level of a local community that needs to take care of a number of refugees that were assigned to them, for example, by the redistribution mechanisms (as in [6]). In particular, the community is modelled as an undirected graph with vertices being the houses of the community. Then, two houses are connected by an edge if they are neighbouring. Some houses are occupied by community members, who also have approval preferences over the numbers of refugees in their neighbourhood. Finally, there is a set of R refugees who need to be housed in empty houses. The solution concept proposed by Knop and Schierreich [7] lies in the selection of exactly R empty houses such that if the refugees are housed in the selected houses, every inhabitant approves the number of refugees in his or her neighbourhood.

[1] https://data.unhcr.org/en/situations

	existence	finding	max SW	PoS
ARH-UB	✓ | ✗	P | NP-hard	NP-hard	$\Theta(n)$
ARH	✗	NP-hard	NP-hard	$\Theta(n)$

Table 1. An overview of our results. In case of two values in a single cell, the first is for swap-stability and the second holds for weak swap-stability.

Our Contribution. The main contribution of our paper lies in the introduction of a new solution concept for refugee housing, which does not suffer from tractability issues of the original variant of Knop and Schierreich [7], while remaining plausible with respect to real-world refugee housing. In particular, we propose the notion of swap-stability that is not only well-known from similar problems [1], but also matches an undesired behaviour of community members. In other words, we are interested in housings where no pair of agents can improve their utility by swapping their position.

First, we show that such swap-stable housing is not guaranteed to exist in all instances of refugee housing, and moreover, that it is NP-hard to decide the existence. We also identify restrictions under which the existence is guaranteed. It is apparent, however, that not all housings are of the same quality. To demonstrate that we use the notion of utilitarian social welfare to distinguish the efficiency of different housings and show that, given a bound ξ, it is NP-hard to decide whether there exists a housing with social welfare at least ξ. Next, we turn our attention to the optimisation variant of the problem and show that, under standard theoretical assumptions, there is no constant-factor polynomial-time approximation algorithm for computing a swap-stable housing maximising social welfare. Finally, we study how much social welfare may be decreased due to the stability requirement, using the notion of the Price of Stability. We summarise our contribution in Table 1.

2 Preliminaries

We consider a set $R = \{r_1, \ldots, r_m\}$ of *refugees*, as well as a set $I = \{h_1, \ldots, h_\ell\}$ of *inhabitants*. Then, the set N of *agents* is the collection of all refugees and inhabitants, that is, $N = R \cup I$. To account for the spatial location of agents we assume a simple, undirected graph $G = (V, E)$, with $|V| \geq |N|$. We will also call V the set of *houses*. Furthermore, for a vertex $v \in V$, let the set $N(v) = \{v' \in V \mid \{v, v'\} \in E\}$ be the *neighbourhood* of v.

Moreover, let $\iota \colon I \to V$ be an injective *assignment* of houses to inhabitants. Then we denote as V_I the set of all houses assigned to inhabitants, i.e., $V_I = \{v \in V \mid \iota(h) = v \text{ for some } h \in I\}$. Also, for each inhabitant $h \in I$, let $U_h = \{N(\iota(h)) \setminus V_I\}$ be the set of houses in the neighbourhood of h that are not occupied by any inhabitant. Furthermore, let $V_U = V \setminus V_I$ be the set of houses not occupied by inhabitants.

We denote the *housing* as an injective function $\pi \colon R \to V_U$. Then we denote as an *allocation* α the assignment of houses to both inhabitants and refugees, that is, $\alpha = \iota \cup \pi$. Finally, for agent $a \in N$, we denote by $\alpha(a)$ the house $\iota(a)$ if a is an inhabitant, or $\pi(a)$ if a is a refugee.

Solution Concepts. In this paper, we combine two solution concepts that arise from different areas of mathematics and computer science. We are primarily interested in finding allocations in which agents do not have the incentive to swap their positions. This corresponds to a game-theoretic notion of *swap stability*. We further extend our investigations using the notion of *approval scores*, known from voting theory to describe outcomes that maximise utilitarian social welfare.

Formally, for an allocation α, a *utility* $\mathrm{u}_a(\alpha)$ for each agent $a \in N$ is equal to 1 if a approves their neighbourhood and 0 otherwise. Then we say that a pair of agents $i, j \in N$ for an allocation α is a *blocking pair* if, for the allocation α' where $\alpha'(i) = \alpha(j), \alpha'(j) = \alpha(i)$, while for every agent $l \notin \{i, j\}$, $\alpha'(l) = \alpha(l)$, we have $\mathrm{u}_i(\alpha') > \mathrm{u}_i(\alpha)$ and $\mathrm{u}_j(\alpha') > \mathrm{u}_j(\alpha)$. Similarly, i, j is a *weakly blocking pair* if $\mathrm{u}_i(\alpha') > \mathrm{u}_i(\alpha)$ and $\mathrm{u}_j(\alpha') \geq \mathrm{u}_j(\alpha)$, or $\mathrm{u}_i(\alpha') \geq \mathrm{u}_i(\alpha)$ and $\mathrm{u}_j(\alpha') > \mathrm{u}_j(\alpha)$ Subsequently, we say that α is *(weakly) swap-stable* for a set of agents $X \subseteq N$, if X does not contain a (weakly) blocking pair. Furthermore, the total *social welfare* of an allocation α is then simply the sum of the utilities of all agents, that is, $\mathrm{SW}(\alpha) = \sum_{a \in N} \mathrm{u}_a(\alpha)$. In our setting with approval preferences, this is in accordance with the number of satisfied agents.

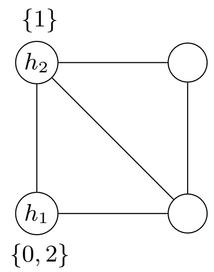
Price of Stability. We will be further interested in how social welfare might be decreased due to achieving swap-stability. Here, the *price of stability* (PoS) [3] is the ratio of social welfare in an optimal allocation and in the worst swap-stable allocation. We denote as $\alpha_{\max}$ the allocation maximising social welfare and as $\alpha^{\textit{stable}}_{\min}$ a swap-stable allocation minimising social welfare.

Definition 1. *For an instance $\mathcal{I}$ with a set of refugees R, a set of inhabitants I, a topology G, and an assignment ι, the* price of stability *(PoS) of $\mathcal{I}$ is defined as*

$$\mathrm{PoS}(\mathcal{I}) = \frac{\mathrm{SW}(\alpha_{\max})}{\mathrm{SW}(\alpha^{\textit{stable}}_{\min})}.$$

ANONYMOUS REFUGEE HOUSING Problem. Here, we define the computational problem we consider. Formally, the input of the ANONYMOUS REFUGEE HOUSING problem (ARH) consists of a topology G, a set of inhabitants I together with their assignment ι to the houses of the topology, a set of refugees R, and a set $A_h \subseteq \{1, \ldots, |R|\}$ of the approved numbers of refugees in the neighbourhood for each inhabitant $h \in I$. The goal is then to decide whether there exists a housing π that is swap-stable. We note that this might not always be the case, as shown in the following instance.

Example 2.1. *Suppose that the topology is the following*



and we have two refugees r_1 and r_2 to house. There are two possible housings. Regardless of the particular housing, the inhabitants have an incentive to swap their position, as both will benefit from it.

Due to the complexity of the setting given above, Knop and Schierreich [7] also proposed a special case of ANONYMOUS REFUGEE HOUSING where every inhabitant gives only an upper limit $\mathrm{ub}(h)$ on the number of refugees in his neighbourhood, that is, for every $h \in I$, we have $A_h = \{0, \ldots, \mathrm{ub}(h)\}$. Schierreich [9] further investigated the computational complexity of this variant in the standard model. We also build upon this simplification, and we call this variant ANONYMOUS REFUGEE HOUSING WITH UPPER-BOUNDS (ARH-UB for short).

3 Finding Swap-Stable Housings

In this section, we focus on the existence guarantees for (weakly) swap-stable housings. We further study the complexity of finding such housings. Perhaps surprisingly, all of our results in this section are positive. As we show, (weakly) swap-stable housings always exist and can be found in polynomial time. We start our study with the simpler case of ANONYMOUS REFUGEE HOUSING WITH UPPER-BOUNDS and swap-stable housings.

Theorem 1. *For every instance of the* ARH-UB *problem, every housing π is swap-stable.*

Proof. Let π be housing and suppose that there is a pair of inhabitants h_1 and h_2 that form a blocking pair. If the number of refugees assigned to the neighbourhood of h_1 is the same as the number of refugees assigned to the neighbourhood of h_2, then h_1 and h_2 cannot be a blocking pair as the utility after the swap remains the same. Since the utility function is binary, $\mathrm{u}_{h_1}(\alpha) = \mathrm{u}_{h_2}(\alpha) = 0$. Without loss of generality, let h_1 be the inhabitant with a higher number of refugees in the neighbourhood. After the swap, the number of refugees in the neighbourhood of h_2 can only increase and, therefore, the h_2's utility cannot increase, which is a contradiction with h_1 and h_2 being a blocking pair. $\square$

If we weaken the notion of swap-stability, it no longer holds that every housing is stable – suppose, e.g., an instance where the topology is a path with four vertices $v_1, \ldots, v_4$, two inhabitants h_1 and h_2 are assigned to vertices v_1 and v_3, respectively, $\mathrm{ub}(h_1) = 2$, $\mathrm{ub}(h_2) = 1$, and $R = 2$. Most notably, in this particular instance no housing is swap-stable, which leads to the following result.

Theorem 2. *There is an instance of the* ARH-UB *problem without a weakly swap-stable housing and it is* NP*-hard to decide whether a weakly swap-stable housing exists.*

As the last result of this section, we turn our attention to the case of general ANONYMOUS REFUGEE HOUSING and show that similar arguments as in Theorem 2 can also be extended for general anonymous preferences.

Theorem 3. *There is an instance of the* ARH *problem without a (weakly) swap-stable housing and it is* NP*-hard to decide whether a (weakly) swap-stable housing exists.*

4 Maximising Social Welfare and Price of Stability

Here, we investigate the efficiency of swap-stable housing through the lens of utilitarian social welfare. Already our first result paints an interesting dichotomy between the cases with and without the requirement on the quality of the outcome (cf. Theorem 2). First, we demonstrate that finding a swap-stable housing optimising social welfare is computationally hard. Note that since ARH-UB is a special case of ARH, the computational hardness results directly carry over to the more general setting.

Theorem 4. *Given an integer value ξ, it is* NP*-complete to decide whether* ARH-UB *admits a (weakly) swap-stable housing with social welfare at least ξ.*

We note that an input Theorem 4 might require all of the inhabitants to be satisfied, which potentially contributes to the complexity of this problem. In our next result we show that relaxing this condition by a limited factor does not make the problem tractable. In particular, we show that for any $0 < q \leq 1$, there is no polynomial-time algorithm that returns a swap-stable allocation with social welfare at least $\frac{1}{q} \cdot \mathrm{OPT}$, where OPT is the maximum social welfare over all swap-stable housings for the given instance (we call the corresponding computational problem MAX-SW-ARH-UB).

Theorem 5. *Unless* P = NP*, there is no polynomial time q-approximation algorithm for the* MAX-SW-ARH-UB *problem for any* $0 < q \leq 1$.

On a more positive note, in our next results, we show that for certain restrictions of the topology, the computation of an assignment maximising social welfare is tractable. The algorithm is based on surprisingly non-trivial dynamic programming.

Theorem 6. *If the topology is a forest, we can find a swap-stable housing that maximises social welfare in polynomial time.*

The final result of our paper shows how much we can lose in social welfare if we additionally require the housing to be stable.

Theorem 7. *There is an instance $\mathcal{I}$ of the* ARH-UB *problem such that the PoS of every (weakly) swap-stable housing is $\Theta(n)$.*

Proof sketch. Let the topology be a disjoint union of a star with $n-1$ leaves and a single edge $\{u, v\}$. Moreover, let the leaves of the star be occupied by inhabitants with the upper-bound equal to zero, let v be occupied by an inhabitant g with the upper-bound equal to one, and let there be a single refugee to house. If we house the refugee in the centre of the star, the social welfare will be equal to 1 as every inhabitant occupying the leaves is unsatisfied, while the inhabitant g is clearly satisfied. On the other hand, if we house the refugee on the vertex u, then the social welfare is equal to $|I|$. Due to Theorem 1, both housings are swap-stable and, therefore, the PoS is indeed $\Theta(n)$. □

5 Other Results and Future Work

There are multiple avenues for further research, and we have already begun exploration of some of them. Most notably, in this paper, we have only studied the case of anonymous preferences. However, Knop and Schierreich [7] also introduced two other variants of preferences in the context of refugee housing, i.e., *hedonic* preferences, where the particular identity of each agent matters and the preferences of all agents consist of approved subsets of agents of the opposite type, and *diversity* preferences. There, agents are additionally partitioned into k classes and every agent has preferences over combinations of fractions of each class in the neighbourhood. In our ongoing work, we investigate the existence of stable allocations in the context of these preferences, depending on the operations the agents are allowed to perform in order to improve their utility, focusing on computational complexity analysis.

However, some directions are still waiting to be explored. For example, does stable housing always exist if we allow agents to move or jump to empty houses of the topology? Or does the complexity picture of the problem change if we allow two agents to swap only if they are not too far from each other? Finally, is swap-stable housing still guaranteed to exist if we aim for more expressive models of preferences such as ordinal or cardinal preferences?

Acknowledgements

ŠS was supported by the Czech Science Foundation Grant No. 22-19557S and by the Grant Agency of the Czech Technical University in Prague, grant No. SGS23/205/OHK3/3T/18. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 101002854).

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